

# On representation theorem of sublinear expectation related to $G$ -Lévy process and paths of $G$ -Lévy process

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## Abstract

In this paper, we are concerned with the representation of an important sublinear expectation  $\mathbb{E}^G[\cdot]$  under which framework a new stochastic process  $G$ -Lévy process has been introduced. We show the existence of a weakly compact family of probability measures  $\mathcal{P}$  to give the representation by using two different methods.

*Key words:* sublinear expectation,  $G$ -Lévy process, càdlàg paths.

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## 1. Introduction

In 2006, [12] introduced the notion of  $G$ -Brownian motion and  $G$ -expectation by using a nonlinear partial differential equation called  $G$ -heat equation. Under this fully nonlinear  $G$ -expectation which serves as a tool to measure risk and model uncertainty, corresponding theory of stochastic calculus is further developed, such as a new type of Itô formula, the existence and uniqueness for stochastic differential equation driven by  $G$ -Brownian motion and martingale representation theorem for  $G$ -expectation, etc. (see [13], [14], [17]).

In [6] and [10], it has been proved that a  $G$ -expectation admits a representation with respect to a weakly compact set of probability measures. Recently, Hu and Peng have introduced in [9] a new stochastic process called  $G$ -Lévy process under a framework of sublinear expectation  $\mathbb{E}^G$  which derives from an integro-partial differential equation. In this paper, motivated by both two methods established respectively in [10] and Section 4.1 of [6], we use both the elementary representation of sublinear expectation and the

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stochastic control method to find a weakly relatively compact family of probability measures  $\{P_\theta : \theta \in \Theta\}$  which gives the following representation :

$$\mathbb{E}^G[X] = \sup_{\theta \in \Theta} \int_{\Omega} X(\omega) dP_\theta(\omega),$$

where the state space  $\Omega$  in our study is  $D([0, \infty), \mathbb{R}^d)$ .

This paper is organized as follows : in Section 2, we review some important notions and results of sublinear expectations,  $G$ -Lévy process, capacity and function space related to an upper-expectation. In Section 3, we use the elementary representation of sublinear expectation by a family of finitely additive linear expectations  $\{E_\lambda : \lambda \in \Lambda\}$  to find the desired weakly relatively compact family of probability measures  $\{P_\theta : \theta \in \Theta\}$ . A concrete weakly relatively compact family of probability measures is constructed through the method of optimal stochastic control in Section 4. In Section 5, we give some elementary characterizations of  $L_{ip}(\Omega)$ . A generalized Kolmogorov-Chentsov's criterion for weak relative compactness has also been presented in the Appendix.

## 2. Basic settings

For a given positive integer  $n$ , we will denote by  $\langle x, y \rangle$  the scalar product of  $x, y \in \mathbb{R}^n$  and by  $|x| = \langle x, x \rangle^{\frac{1}{2}}$  the Euclidean norm of  $x$ .

Let  $\Omega$  be a given set and let  $\mathcal{H}$  be a linear space of real-valued functions defined on  $\Omega$  such that for each integer  $n$ , if  $X_1, X_2, \dots, X_n \in \mathcal{H}$ , then  $\varphi(X_1, X_2, \dots, X_n) \in \mathcal{H}$  for each  $\varphi \in C_{b,Lip}(\mathbb{R}^n)$ , where  $C_{b,Lip}(\mathbb{R}^n)$  denotes the space of bounded and Lipschitz functions on  $\mathbb{R}^n$ .  $\mathcal{H}$  is usually considered as a space of random variables.

In the following we will present some preliminaries in the theory of sublinear expectation and the related  $G$ -Lévy process. More details of this section can be found in [9].

**Definition 2.1** A functional  $\hat{\mathbb{E}} : \mathcal{H} \rightarrow \mathbb{R}$  is called a sublinear expectation if it satisfies the following properties : for all  $X, Y \in \mathcal{H}$ , we have

- (a) *Monotonicity* : if  $X \geq Y$ , then  $\hat{\mathbb{E}}[X] \geq \hat{\mathbb{E}}[Y]$ .
- (b) *Constant preserving* :  $\hat{\mathbb{E}}[c] = c, \forall c \in \mathbb{R}$ .
- (c) *Sub-additivity* :  $\hat{\mathbb{E}}[X] - \hat{\mathbb{E}}[Y] \leq \hat{\mathbb{E}}[X - Y]$ .
- (d) *Positive homogeneity* :  $\hat{\mathbb{E}}[\lambda X] = \lambda \hat{\mathbb{E}}[X], \forall \lambda \geq 0$ .

We call the triple  $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$  a sublinear expectation space similar to the probability space  $(\Omega, \mathcal{F}, P)$ .

**Definition 2.2** Let  $X_1$  and  $X_2$  be two  $n$ -dimensional random vectors defined respectively on the sublinear expectation spaces  $(\Omega_1, \mathcal{H}_1, \hat{\mathbb{E}}_1)$  and  $(\Omega_2, \mathcal{H}_2, \hat{\mathbb{E}}_2)$ . They are called identically distributed if  $\hat{\mathbb{E}}_1[\varphi(X_1)] = \hat{\mathbb{E}}_2[\varphi(X_2)]$  for all  $\varphi \in C_{b,Lip}(\mathbb{R}^n)$ , denoted by  $X_1 \stackrel{D}{=} X_2$ .

**Definition 2.3** On a sublinear expectation space  $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ , an  $n$ -dimensional random vector  $Y$  is said to be independent of another  $m$ -dimensional random vector  $X$ , if for each test function  $\varphi \in C_{b,Lip}(\mathbb{R}^m \times \mathbb{R}^n)$ , we have

$$\hat{\mathbb{E}}[\varphi(X, Y)] = \hat{\mathbb{E}}[\hat{\mathbb{E}}[\varphi(x, Y)]_{x=X}].$$

## 2.1. $G$ -Lévy process and Sublinear expectation related to $G$ -Lévy process

**Definition 2.4 (Lévy process)** Let  $X = (X_t)_{t \geq 0}$  be a  $d$ -dimensional càdlàg process defined on a sublinear expectation space  $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ . We say that  $X$  is a Lévy process if :

- (a)  $X_0 = 0$ ;
- (b) Independent increments : for each  $t, s > 0$ , the increments  $X_{t+s} - X_t$  is independent of  $(X_{t_1}, X_{t_2}, \dots, X_{t_n})$ , for each  $n \in \mathbb{N}$  and  $0 \leq t_1 \leq \dots \leq t_n \leq t$ ;
- (c) Stationary increments : the distribution of  $X_{t+s} - X_t$  does not depend on  $t$ .

**Definition 2.5 ( $G$ -Lévy process)** A  $d$ -dimensional process  $(X_t)_{t \geq 0}$  on a sublinear expectation space  $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$  is called a  $G$ -Lévy process if :

- (a)  $X$  satisfies (a)-(c) in Definition. 2.4 ;
- (b) for each  $t \geq 0$ , there exists a decomposition  $X_t = X_t^c + X_t^d$ ;
- (c)  $(X_t^c, X_t^d)_{t \geq 0}$  is a 2d-dimensional Lévy process satisfying

$$\lim_{t \downarrow 0} \hat{\mathbb{E}}[|X_t^c|^3] t^{-1} = 0; \quad \hat{\mathbb{E}}[|X_t^d|] \leq Ct \text{ for } t \geq 0, \quad (1)$$

where  $C$  is a constant.

**Remark 1** We know that in classical linear expectation case, assumption (b) of  $(X_t)_{t \geq 0}$  obviously holds by the Lévy-Itô decomposition, where  $(X_t^c)_{t \geq 0}$  and  $(X_t^d)_{t \geq 0}$  represent the continuous part and jump part respectively. By assumption (1) on  $(X_t^c)_{t \geq 0}$  and  $(X_t^d)_{t \geq 0}$ , we know that  $(X_t^c)_{t \geq 0}$  is a generalized  $G$ -Brownian motion, and  $(X_t^d)_{t \geq 0}$  is of finite variation.

**Theorem 2.6** Let  $(X_t)_{t \geq 0}$  be a  $d$ -dimensional  $G$ -Lévy process with the decomposition  $X_t = X_t^c + X_t^d$ . For each  $\varphi \in C_{b,Lip}(\mathbb{R}^n)$ ,  $u(t, x) := \hat{\mathbb{E}}[\varphi(x + X_t)]$  is a viscosity solution of the following equation :

$$\partial_t u(t, x) - G_X[u(t, x + \cdot) - u(t, x)] = 0, u(0, x) = \varphi(x),$$

where  $G_X[f(\cdot)]$  is a nonlocal operator defined by

$$G_X[f(\cdot)] := \lim_{\delta \downarrow 0} \hat{\mathbb{E}}[f(X_\delta)] \delta^{-1} \text{ for } f \in C_b^3(\mathbb{R}^d) \text{ with } f(0) = 0.$$

Now we will consider a particular sublinear expectation related to  $G$ -Lévy process, and throughout this paper our study will be worked under such a sublinear expectation. Let  $\Omega = D_0([0, \infty), \mathbb{R}^d)$  denote the space of all  $\mathbb{R}^d$ -valued càdlàg paths  $(\omega_t)_{t \geq 0}$  with  $\omega_0 = 0$ , equipped with the distance (introduced in [3])

$$d_\infty^\circ(\omega^1, \omega^2) := \sum_{i=1}^{\infty} 2^{-i} [d_i^\circ(\omega_i^1, \omega_i^2) \wedge 1],$$

where  $d_t^\circ$  is a metric defined on  $D_0([0, t], \mathbb{R}^d)$ .  $D_0([0, t], \mathbb{R}^d)$  is complete and separable under  $d_t^\circ$ , hence is a Polish space.  $D_0([0, \infty), \mathbb{R}^d)$  is also a Polish space under the Skorohod topology  $d_\infty^\circ$ . Let  $\mathcal{B}(\Omega)$  denote the  $\sigma$ -algebra generated by all open sets. We will consider the canonical process  $B_t(\omega) = \omega_t$  for  $\omega \in \Omega$ ,  $t \geq 0$ . We introduce the space of finite dimensional cylinder random variables : for each fixed  $T \geq 0$ , we denote  $\Omega_T = \{\omega_{\cdot \wedge T} : \omega \in \Omega\}$  and set

$$L_{ip}(\Omega_T) := \{\varphi(B_{t_1 \wedge T}, B_{t_2 \wedge T}, \dots, B_{t_n \wedge T}) : n \in \mathbb{N}, t_1, t_2, \dots, t_n \in [0, \infty), \varphi \in C_{b,Lip}(\mathbb{R}^{d \times n})\};$$

$$L_{ip}(\Omega) := \{\varphi(B_{t_1}, B_{t_2}, \dots, B_{t_n}) : n \in \mathbb{N}, t_1, t_2, \dots, t_n \in [0, \infty), \varphi \in C_{b,Lip}(\mathbb{R}^{d \times n})\}.$$

It is clear that for  $t \leq T$ ,  $L_{ip}(\Omega_t) \subseteq L_{ip}(\Omega_T)$  and  $L_{ip}(\Omega) = \cup_{n=1}^{\infty} L_{ip}(\Omega_n)$ .

A sublinear expectation  $\hat{\mathbb{E}}[\cdot]$  defined on  $L_{ip}(\Omega)$  through the following procedure is a sublinear expectation related to the  $G$ -Lévy process :

Step 1. For each  $\xi \in L_{ip}(\Omega)$  with the form  $\xi = \varphi(B_{t+s} - B_t)$ ,  $t, s \geq 0$  and  $\varphi \in C_{b,Lip}(\mathbb{R}^n)$ . We define  $\hat{\mathbb{E}}[\xi] = u(s, 0)$ , where  $u$  is a viscosity solution of the following integro-partial differential equation :

$$\frac{\partial u}{\partial t} = G(u, Du, D^2u), \quad \text{on } (t, x) \in [0, T] \times \mathbb{R}^d \quad (2)$$

with the initial condition  $u(0, x) = \varphi(x)$ , where

$$G(u, p, A) = \sup_{(\nu, q, Q) \in \mathcal{U}} \left\{ \int_{\mathbb{R}^d \setminus \{0\}} (u(t, x+z) - u(t, x)) \nu(dz) + \langle p, q \rangle + \frac{1}{2} \text{tr}[AQQ^T] \right\},$$

and  $\mathcal{U}$  represents  $G_X$ .

Step 2. For each  $\xi \in L_{ip}(\Omega)$ , there exists a  $\psi \in C_{b,Lip}(\mathbb{R}^{d \times m})$  such that  $\xi = \psi(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \dots, B_{t_m} - B_{t_{m-1}})$  for some  $t_1 < t_2 < \dots < t_m$ . Then  $\hat{\mathbb{E}}[\xi]$  is defined by  $\psi_m$  via the following procedure :

$$\begin{aligned} \psi_1(x_1, \dots, x_{m-1}) &= \hat{\mathbb{E}}(\psi(x_1, \dots, x_{m-1}, B_{t_m} - B_{t_{m-1}})); \\ \psi_2(x_1, \dots, x_{m-2}) &= \hat{\mathbb{E}}(\psi_1(x_1, \dots, x_{m-2}, B_{t_{m-1}} - B_{t_{m-2}})); \\ &\vdots \\ \psi_{m-1}(x_1) &= \hat{\mathbb{E}}(\psi_{m-2}(x_1, B_{t_2} - B_{t_1})); \\ \psi_m &= \hat{\mathbb{E}}(\psi_{m-1}(B_{t_1})). \end{aligned}$$

The related conditional expectation of  $\xi$  under  $\Omega_{t_i}$  is defined by

$$\begin{aligned} \hat{\mathbb{E}}[\xi | \Omega_{t_j}] &= \hat{\mathbb{E}}[\psi(B_{t_1}, B_{t_2} - B_{t_1}, \dots, B_{t_m} - B_{t_{m-1}}) | \Omega_{t_j}] \\ &= \psi_{m-j}(B_{t_1}, \dots, B_{t_j} - B_{t_{j-1}}). \end{aligned}$$

It can be proved that the canonical process  $(B_t)_{t \geq 0}$  is a  $G$ -Lévy process, and  $\hat{\mathbb{E}}[\cdot]$  consistently defines a sublinear expectation on  $L_{ip}(\Omega)$ . Then for  $p \geq 1$ , the topological completion of  $L_{ip}(\Omega_T)$  (resp.  $L_{ip}(\Omega)$ ) under the Banach norm  $\|\cdot\|_p := \hat{\mathbb{E}}[|\cdot|^p]^{\frac{1}{p}}$  is denoted by  $L_G^p(\Omega_T)$  (resp.  $L_G^p(\Omega)$ ).  $\hat{\mathbb{E}}[\cdot]$  can be extended continuously and uniquely from  $L_{ip}(\Omega_T)$  (resp.  $L_{ip}(\Omega)$ ) into  $L_G^p(\Omega_T)$  (resp.  $L_G^p(\Omega)$ ), and it is still a sublinear expectation on the corresponding completed space.

## 2.2. Capacity Associated to an Upper Probability and related functional spaces

Let  $\Omega$  be a complete separable metric space equipped with the distance  $d$ ,  $\mathcal{B}(\Omega)$  the Borel  $\sigma$ -algebra of  $\Omega$  and  $\mathcal{M}$  the collection of all probability measures on  $(\Omega, \mathcal{B}(\Omega))$ . We denote by  $L^0(\Omega)$  the space of all  $\mathcal{B}(\Omega)$ -measurable real functions and  $C_b(\Omega)$  all bounded continuous functions. For a given subset  $\mathcal{P} \subseteq \mathcal{M}$ , we denote

$$c(A) := \sup_{P \in \mathcal{P}} P(A), \quad A \in \mathcal{B}(\Omega).$$

It is easy to verify that  $c(\cdot)$  satisfies the following theorem.

**Theorem 2.7** *The set function  $c(\cdot)$  is a Choquet capacity, i.e. (see [4], [5])*

- (a)  $0 \leq c(A) \leq 1, \forall A \subset \Omega$ .
- (b) If  $A \subset B$ , then  $c(A) \leq c(B)$ .
- (c) If  $(A_n)_{n=1}^\infty$  is a sequence in  $\mathcal{B}(\Omega)$ , then  $c(\bigcup A_n) \leq \sum c(A_n)$ .
- (d) If  $(A_n)_{n=1}^\infty$  is an increasing sequence in  $\mathcal{B}(\Omega) : A_n \uparrow A = \bigcup A_n$ , then  $c(\bigcup A_n) = \lim_{n \rightarrow \infty} c(A_n)$ .

**Definition 2.8** *We say that a set  $A$  is polar if  $c(A) = 0$  and a property holds "quasi-surely" (q.s.) if it holds outside a polar set. In other words,  $A \in \mathcal{B}(\Omega)$  is polar if and only if  $P(A) = 0$  for any  $P \in \mathcal{P}$ .*

By applying the Borel-Cantelli Lemma we could get immediately the following lemma.

**Lemma 2.9** *Let  $(A_n)_{n \in \mathbb{N}}$  be a sequence of Borel sets such that*

$$\sum_{n=1}^{\infty} c(A_n) < \infty.$$

*Then  $\limsup_{n \rightarrow \infty} A_n$  is polar.*

The upper expectation (see [11]) of  $\mathcal{P}$  is defined as follows : for each  $X \in L^0(\Omega)$  such that  $E_P[X]$  exists for each  $P \in \mathcal{P}$ ,

$$\mathbb{E}[X] = \mathbb{E}^{\mathcal{P}}[X] := \sup_{P \in \mathcal{P}} E_P[X].$$

If for  $p > 0$ , we denote

- $\mathcal{L}^p := \{X \in L^0(\Omega) : \mathbb{E}[|X|^p] = \sup_{P \in \mathcal{P}} E_P[|X|^p] < \infty\}$ ;
- $\mathcal{N}^p := \{X \in L^0(\Omega) : \mathbb{E}[|X|^p] = 0\}$ ;
- $\mathcal{N} := \{X \in L^0(\Omega) : X = 0, c - q.s.\}$ .

It is seen that  $\mathcal{L}^p$  and  $\mathcal{N}^p$  are linear spaces and  $\mathcal{N}^p = \mathcal{N}$ , for  $p > 0$ . All of the following definitions and propositions can be found in [6], and the proofs are omitted here.

**Proposition 2.1** *For each  $p \geq 1$ ,  $\mathbb{L}^p := \mathcal{L}^p / \mathcal{N}$  is a Banach space under the norm  $\|X\|_p := (\mathbb{E}[|X|^p])^{1/p}$ ; for each  $p < 1$ ,  $\mathbb{L}^p$  is a complete metric space under the distance  $d(X, Y) := \mathbb{E}[|X - Y|^p]$ .*

With respect to the distance defined on  $\mathbb{L}^p$ ,  $p > 0$ , we denote :

- $\mathbb{L}_c^p$  the completion of  $C_b(\Omega)$ ;
- $L_G^p(\Omega)$  the completion of  $L_{ip}(\Omega)$ .

**Definition 2.10** *A mapping  $X$  on  $\Omega$  with values in a topological space is said to be quasi-continuous if for any  $\varepsilon > 0$ , there exists an open subset  $O$  with  $c(O) < \varepsilon$  such that  $X|_{O^c}$  is continuous.*

**Definition 2.11** *We say that  $X : \Omega \rightarrow \mathbb{R}$  has a quasi-continuous version if there exists a quasi-continuous function  $Y : \Omega \rightarrow \mathbb{R}$  with  $X = Y$  q.s..*

**Proposition 2.2** *For each  $p > 0$ ,*

$$\mathbb{L}_c^p = \{X \in \mathbb{L}^p : X \text{ has a quasi-continuous version, } \lim_{n \rightarrow \infty} \mathbb{E}[|X|^p I_{\{|X| > n\}}] = 0\}.$$

**Proposition 2.3** *Let  $\{P_n\}_{n=1}^\infty \subset \mathcal{P}$  converge weakly to  $P \in \mathcal{P}$ . Then for each  $X \in \mathbb{L}_c^1$ , we have  $E_{P_n}[X] \rightarrow E_P[X]$  as  $n \rightarrow \infty$ .*

### 3. Representation of the sublinear expectation related to $G$ -Lévy process as an upper-Expectation

Let  $\bar{\Omega} = (\mathbb{R}^d)^{[0, \infty)}$  denote the space of all  $\mathbb{R}^d$ -valued functions  $(\bar{\omega}_t)_{t \geq 0}$  and  $\mathcal{B}(\bar{\Omega})$  denote the  $\sigma$ -algebra generated by all finite dimensional cylinder sets. The corresponding canonical process  $\bar{B}_t(\bar{\omega}) = \bar{\omega}_t$  for  $\bar{\omega} \in \bar{\Omega}$ ,  $t \geq 0$ . The space of Lipschitzian cylinder functions on  $\bar{\Omega}_T$  is denoted by

$$L_{ip}(\bar{\Omega}) := \{\varphi(\bar{B}_{t_1}, \bar{B}_{t_2}, \dots, \bar{B}_{t_n}) : n \in \mathbb{N}, t_1, t_2, \dots, t_n \in [0, \infty), \varphi \in C_{b,Lip}(\mathbb{R}^{d \times n})\}$$

and

$$L_{ip}(\bar{\Omega}_T) := \{\varphi(\bar{B}_{t_1 \wedge T}, \bar{B}_{t_2 \wedge T}, \dots, \bar{B}_{t_n \wedge T}) : n \in \mathbb{N}, t_1, t_2, \dots, t_n \in [0, \infty), \varphi \in C_{b,Lip}(\mathbb{R}^{d \times n})\}.$$

Following the same procedure as the construction of  $\hat{\mathbb{E}}[\cdot]$ , we can also construct a sublinear expectation  $\bar{\mathbb{E}}$  on  $(\bar{\Omega}, L_{ip}(\bar{\Omega}))$  such that  $(\bar{B}_t(\bar{\omega}))_{t \geq 0}$  is also a  $G$ -Lévy process.

The following lemmas can be found in [10] and [15].

**Lemma 3.1** *Let  $\hat{\mathbb{E}}$  be a sublinear functional defined on a linear space  $\mathcal{H}$ , i.e. (c) and (d) of Definition 2.1 hold for  $\hat{\mathbb{E}}$ . Then there exists a family  $\mathcal{Q} = \{E_\theta : \theta \in \Theta\}$  of linear functionals defined on  $\mathcal{H}$  such that*

$$\hat{\mathbb{E}}[X] := \sup_{\theta \in \Theta} E_\theta[X], \text{ for } X \in \mathcal{H}.$$

*and such that, for each  $X \in \mathcal{H}$ , there exists a  $\theta \in \Theta$  such that  $\hat{\mathbb{E}}[X] = E_\theta[X]$ . Moreover if  $\hat{\mathbb{E}}$  is a sublinear functional defined on a linear space  $\mathcal{H}$  of functions on  $\Omega$  such that (a) of Definition 2.1 holds (resp. (a),(b) hold) for  $\hat{\mathbb{E}}$ , then (a) also holds (resp. (a),(b) hold) for  $E_\theta$ ,  $\theta \in \Theta$ .*

**Lemma 3.2** *Let  $0 \leq t_1 < t_2 < \dots < t_m < \infty$  and  $\{\varphi_n\}_{n=1}^\infty \subset C_{b,Lip}(\mathbb{R}^{d \times m})$  satisfy  $\varphi_n \downarrow 0$ , then  $\bar{\mathbb{E}}[\varphi_n(\bar{B}_{t_1}, \bar{B}_{t_2}, \dots, \bar{B}_{t_m})] \downarrow 0$ .*

**Lemma 3.3** *Let  $E$  be a finitely additive linear expectation dominated by  $\bar{\mathbb{E}}$  on  $L_{ip}(\bar{\Omega})$ , then there exists a unique probability measure  $Q$  on  $(\bar{\Omega}, \mathcal{B}(\bar{\Omega}))$  such that  $E[X] = E_Q[X]$  for each  $X \in L_{ip}(\bar{\Omega})$ .*

**Remark 2** *This is a direct result of Daniell-Stone's theorem and Kolmogorov's consistent theorem with the help of above lemma.*

By Lemma 3.1 and Lemma 3.3, it is easy to get the next result crucial for our following discussions.

**Lemma 3.4** *There exists a family of probability measures  $\mathcal{P}_e$  on  $(\bar{\Omega}, \mathcal{B}(\bar{\Omega}))$  such that*

$$\bar{\mathbb{E}}[X] = \max_{Q \in \mathcal{P}_e} E_Q[X], \forall X \in L_{ip}(\bar{\Omega}).$$

We denote the associated capacity to  $\mathcal{P}_e$  by

$$\tilde{c}(A) := \sup_{Q \in \mathcal{P}_e} Q(A), \quad A \in \mathcal{B}(\bar{\Omega}).$$

and related upper expectation for each  $\mathcal{B}(\bar{\Omega})$ -measurable real function  $X$  which makes the following definition meaningful,

$$\tilde{\mathbb{E}}[X] := \sup_{Q \in \mathcal{P}_e} E_Q[X].$$

**Definition 3.5** *Let  $I$  be a set of indices,  $(X_t)_{t \in I}$  and  $(Y_t)_{t \in I}$  be two processes indexed by  $I$ .  $Y$  is said to be a quasi-modification of  $X$  in for all  $t \in I$ ,  $X_t = Y_t$  q.s..*

**Definition 3.6** *Let  $\varepsilon$  be a positive number. A function  $y = X(t)$  is said to have no fewer than  $m$   $\varepsilon$ -oscillations on a closed interval  $[a, b]$  if there exist points  $a \leq t_0 < t_1 < \dots < t_m \leq b$ , such that  $|X_{t_{k-1}} - X_{t_k}| > \varepsilon$  for  $k = 1, 2, \dots, m$ .*

**Remark 3** *We know that for a function  $y = X(t)$  to be càdlàg on a interval  $[a, b]$ , it is necessary and sufficient that for arbitrary  $\varepsilon > 0$ , it has only finitely many  $\varepsilon$ -oscillations on  $[a, b]$ .*

**Theorem 3.7 (Kolmogorov-Chentsov's Criterion)** *Let  $(X_t)_{t \in [0,1]}$  be a stochastic process such that for all  $t \in [0,1]$ ,  $X_t$  belongs to  $\mathbb{L}^1$ . If it satisfies the following conditions :*

- (i) *for any  $t \in [0,1]$ , there exists  $a > 0$ , such that  $\lim_{s \rightarrow t} \mathbb{E}[|X_s - X_t|^a] = 0$ ;*
- (ii) *for some  $C, r > 0, p, q \geq 0$  with  $p + q > 0$ , and all  $0 \leq s \leq u \leq t \leq 1$ , it holds that*

$$\mathbb{E}[|X_t - X_u|^p |X_u - X_s|^q] \leq C|t - s|^{1+r}, \quad (3)$$

*then it admits a càdlàg modification.*

*Proof.* It follows from condition (i) that, for any  $\varepsilon > 0$  and  $t \in [0, T]$ ,  $\lim_{s \rightarrow t} c(|X_s - X_t| \geq \varepsilon) = 0$ . Now let us take the set  $J$  of all dyadic numbers belonging to  $[0, 1]$ ,

$$J = \left\{ \frac{i}{2^n}; n \in \mathbb{N}, i \in \{0, 1, \dots, 2^n\} \right\}.$$

It follows from (3) and Chebyshev's inequality that

$$\begin{aligned} c(\{|X_t - X_u| \geq \epsilon_1\} \cap \{|X_u - X_s| \geq \epsilon_2\}) &\leq \mathbb{E} \left[ \frac{|X_t - X_u|^p}{\epsilon_1^p} \frac{|X_u - X_s|^q}{\epsilon_2^q} \right] \\ &\leq \frac{C|t - s|^{1+r}}{\epsilon_1^p \epsilon_2^q}. \end{aligned} \quad (4)$$

Let  $A_{k,n}$  denote the event

$$\left\{ \left| X\left(\frac{k}{2^n}\right) - X\left(\frac{k-1}{2^n}\right) \right| < \varepsilon_n \right\},$$

where

$$\varepsilon_n = \frac{1}{2} C^{1/p+q} \frac{r(n-1)}{2(p+q)} = L\alpha^n,$$

$$L = (2^{r/2}C)^{1/p+q}, \quad \alpha = 2^{-r/2(p+q)} < 1,$$

and

$$B_{kn} = A_{k,n} \cup A_{k+1,n} = \left\{ \left| X\left(\frac{k}{2^n}\right) - X\left(\frac{k-1}{2^n}\right) \right| < \varepsilon_n \right\} \cup \left\{ \left| X\left(\frac{k+1}{2^n}\right) - X\left(\frac{k}{2^n}\right) \right| < \varepsilon_n \right\}.$$

From the inequality (4), we have

$$c(B_{kn}^c) \leq 2^{-(1+r/2)(n-1)}, \quad k = 1, 2, \dots, 2^n - 1,$$

where  $B_{kn}^c$  is the complementary event of  $B_{kn}$ . Let us define

$$D_n := \bigcap_{m=n}^{\infty} \bigcap_{k=1}^{2^m-1} B_{km}, \quad D := \bigcup_{n=1}^{\infty} D_n,$$

with the complementary events

$$D_n^c = \bigcup_{m=n}^{\infty} \bigcup_{k=1}^{2^m-1} B_{km}^c, \quad D^c = \bigcap_{n=1}^{\infty} D_n^c.$$

Then we have

$$c(D_n^c) \leq \sum_{m=n}^{\infty} \sum_{k=1}^{2^m-1} c(B_{km}^c) \leq \sum_{m=n}^{\infty} \sum_{k=1}^{2^m-1} 2^{-(1+r/2)(m-1)} = \frac{2\beta^{n-1}}{1-\beta},$$

where  $\beta = 2^{-r/2} < 1$ , by Lemma 2.9 it follows that  $c(D^c) = 0$ , i.e.  $D^c$  is a polar set. Let us choose a sample function  $X(t)$  for which the event  $A_{k+1,n} \cap D_{n+1}$  occurs. Thanks to the classical Kolmogorov-Chentsov's criterion, we can continue to get the result which is essentially in the spirit of Kolmogorov's criterion, though its proof is much more difficult. We will only outline the proof for the convenience of readers. An interested reader may refer to [8, pp. 159-164] for details. Suppose that the event  $D$  occurs. Then beginning with some  $n_0$ , all the events  $D_n$  for  $n \geq n_0$  occur for the sample function of the process. For arbitrary  $\varepsilon > 0$  we can find an  $n \geq n_0$  such that  $2L\alpha^n/(1-\alpha)^2 < \varepsilon$ . On an arbitrary set of the form

$$J \cap \left[ \frac{k-1}{2^n}, \frac{k+1}{2^n} \right],$$

there can be no more than a single  $\varepsilon$ -oscillation, which yields that the function  $X(t)$  has no more than  $2^n$   $\varepsilon$ -oscillations on  $J$ . Thus outside a polar set, the sample functions of process  $X(t)$  have only finitely many  $\varepsilon$ -oscillations; that is, the process  $X$  has a càdlàg modification.  $\square$

**Remark 4** *This theorem is a generalized Kolmogorov-Chentsov's criterion for càdlàg modification with respect to capacity.*

**Lemma 3.8** *For  $\bar{B} = \{\bar{B}_t : t \geq 0\}$ , there exists a càdlàg modification  $\tilde{B} = \{\tilde{B}_t : t \geq 0\}$  of  $\bar{B}$  (i.e.  $\tilde{c}(\{\bar{B}_t \neq \tilde{B}_t\}) = 0$ , for each  $t \geq 0$ ) such that  $\tilde{B}_0 = 0$ .*

*Proof.* The canonical process  $(\bar{B}_t)$  has a decomposition  $\bar{B}_t = \bar{B}_t^c + \bar{B}_t^d$ . The part  $\bar{B}_t^c$  has a continuous modification  $\tilde{B}_t^c$ . For the other part  $\bar{B}_t^d$ , on the basis of (1) in the definition of  $G$ -Lévy process and Lemma 3.4, we have that, for all  $0 \leq s \leq u \leq t < \infty$ ,

$$\tilde{\mathbb{E}}[|\bar{B}_t^d - \bar{B}_u^d||\bar{B}_u^d - \bar{B}_s^d|] = \tilde{\mathbb{E}}[|\bar{B}_t^d - \bar{B}_u^d||\bar{B}_u^d - \bar{B}_s^d|] \leq C^2|t - s|^2,$$

where the constant  $C$  is a constant. By Theorem 3.7, there exists a càdlàg modification  $\tilde{B}^d$  of  $\bar{B}^d$ . Consequently,  $\tilde{B} = \{\tilde{B}_t : \tilde{B}_t = \tilde{B}_t^c + \tilde{B}_t^d, t \geq 0\}$  is the desired càdlàg modification of  $\bar{B}$ .  $\square$

The family of probability measures  $\mathcal{P}_e$  on  $(\bar{\Omega}, \mathcal{B}(\bar{\Omega}))$  introduces a new family of probability measures  $\mathcal{P}_1 := \{Q \circ \bar{B}^{-1} : Q \in \mathcal{P}_e\}$  on  $(\Omega, \mathcal{B}(\Omega))$ .

**Lemma 3.9** *The family of probability measures  $\mathcal{P}_1$  is tight.*

*Proof.* Since for all  $0 \leq s \leq u \leq t < \infty$ , there exists  $C' > 0$ , such that

$$\tilde{\mathbb{E}}[|\tilde{B}_t^c - \tilde{B}_s^c|^4] = \tilde{\mathbb{E}}[|\bar{B}_t^c - \bar{B}_s^c|^4] \leq C'|t - s|^2,$$

$$\tilde{\mathbb{E}}[|\tilde{B}_t^d - \tilde{B}_u^d||\tilde{B}_u^d - \tilde{B}_s^d|] = \tilde{\mathbb{E}}[|\bar{B}_t^d - \bar{B}_u^d||\bar{B}_u^d - \bar{B}_s^d|] \leq C'|t - s|^2.$$

Due to the generalized Kolmogorov-Chentsov's criterion for tightness (see Theorem A.3) combined with Corollary A.2, this implies the wished statement.  $\square$

By Lemma 3.4 and 3.8, the representation of sublinear expectation related to  $G$ -Lévy process with respect to  $\mathcal{P}_1$  is given by the following theorem.

**Theorem 3.10** *For each monotonic and sublinear function  $G_X[f(\cdot)] : \mathbb{R}^d \mapsto \mathbb{R}$ , where  $f \in C_b^3(\mathbb{R}^d)$  with  $f(0) = 0$ , let  $\mathbb{E}^G$  be the corresponding sublinear expectation on  $(\Omega, L_{ip}(\Omega))$ . Then there exists a relatively compact family of probability measures  $\mathcal{P}_1$  on  $(\Omega, L_{ip}(\Omega))$ , such that*

$$\mathbb{E}^G[X] = \max_{P \in \mathcal{P}_1} E_P[X], \quad \forall X \in L_{ip}(\Omega).$$

#### 4. Representation of $\mathbb{E}^G$ using the stochastic control method

In this section we will construct a family of probability measures on  $\Omega$  for which the upper expectation coincides with the sublinear expectation  $\mathbb{E}^G$  on  $L_{ip}(\Omega)$  through a method of optimal stochastic controls introduced in [6].

Let  $(\Omega, \mathcal{F}, P)$  be a probability space and  $(L_t)_{t \geq 0} = (L_t^i)_{i=1}^d$  a  $d$ -dimensional Lévy process with finite variation in this space. The filtration generated by  $L$  is denoted by

$$\mathcal{F}_t := \sigma\{L_u, 0 \leq u \leq t\} \vee \mathcal{N}, \quad \mathbb{F} := \{\mathcal{F}_t\}_{t \geq 0},$$

where  $\mathcal{N}$  is the collection of  $P$ -null subsets. We also denote, for a fixed  $s \geq 0$ ,

$$\mathcal{F}_t^s := \sigma\{L_{s+u} - L_s, 0 \leq u \leq t\} \vee \mathcal{N}, \quad \mathbb{F}^s := \{\mathcal{F}_t^s\}_{t \geq 0}.$$



Let  $\Theta = (\Theta^c, \Theta^d) = ({}^1\Theta^c, {}^2\Theta^c, \Theta^d)$  be a given bounded and closed subset in  $\mathbb{R}^{d \times 3d}$ . We denote a collection of  $\Theta$ -valued processes on an interval  $[t, T] \subset [0, \infty)$  by  $\mathcal{A}_{t,T}^\Theta := \{\theta = (\theta^c, \theta^d) = ({}^1\theta^c, {}^2\theta^c, \theta^d) : \theta^c \text{ is } \Theta^c\text{-valued } \mathbb{F}\text{-adapted and } \theta^d \text{ is } \Theta^d\text{-valued } \mathbb{F}\text{-predictable}\}$ . For each fixed  $\theta = (\theta^c, \theta^d) \in \mathcal{A}_{t,T}^\Theta$  we denote

$$B_T^{t,\theta} := \int_t^T \theta_s d(L_s^c, L_s^d)^T,$$

where  $L_s^c$  and  $L_s^d$  denote the continuous part and jump part of  $L_s$  respectively, hence  $(B_T^{t,\theta})_{T \geq t}$  is a Lévy stochastic integral. In the following we will prove that, for each  $n = 1, 2, \dots$ , any  $\varphi \in C_{b,Lip}(\mathbb{R}^{d \times n})$  and  $0 \leq t_1, \dots, t_n < \infty$ , the  $\mathbb{E}^G$  defined in [9] can be equivalently defined by

$$\mathbb{E}^G[\varphi(X_{t_1}, X_{t_2} - X_{t_1}, \dots, X_{t_n} - X_{t_{n-1}})] = \sup_{\theta \in \mathcal{A}_{t,T}^\Theta} E_P[\varphi(B_{t_1}^{0,\theta}, B_{t_2}^{t_1,\theta}, \dots, B_{t_n}^{t_{n-1},\theta})].$$

If for each given  $\varphi \in C_{b,Lip}(\mathbb{R}^d)$ , we set  $v(t, x) := \sup_{\theta \in \mathcal{A}_{t,T}^\Theta} E_P[\varphi(x + B_T^{t,\theta})]$ , then we can get the following generalized dynamical programming principle :

**Proposition 4.1** *We have*

$$v(t, x) = \sup_{\theta \in \mathcal{A}_{t,t+h}^\Theta} E_P[v(t+h, x + B_{t+h}^{t,\theta})]. \quad (5)$$

*Proof.* This is a result treated analogous to [6], therefore we omit the proof here.  $\square$

**Lemma 4.2**  *$v$  is bounded by  $\sup |\varphi|$ . It is Lipschitz function in  $x$  and hölder continuous in  $t$ .*

*Proof.* Obviously we know that

$$\sup_{\theta \in \mathcal{A}_{t,t+h}^\Theta} E_P[v(t+h, x + B_{t+h}^{t,\theta}) - v(t+h, x)] = v(t, x) - v(t+h, x),$$

since  $v$  is Lipschitz function in  $x$ , the absolute value of left hand is bounded by

$$C \sup_{\theta \in \mathcal{A}_{t,t+h}^\Theta} E_P[|B_{t+h}^{t,\theta}|] \leq C_1(h + h^{\frac{1}{2}}).$$

We get the result.  $\square$

**Theorem 4.3**  *$v$  is a viscosity solution of the integro-partial differential equation :*

$$\frac{\partial v}{\partial t} + G(v, Dv, D^2v) = 0, \quad \text{on } (t, x) \in [0, T) \times \mathbb{R}^d, \quad (6)$$

$$v(T, x) = \varphi(x),$$

where the function  $G$  is given in (2).

*Proof.* Since  $X_t$  has a decomposition with  $X_t = X_t^c + X_t^d$  and for each  $t \geq 0$ ,  $L_t$  has the following Lévy-Itô decomposition (see e.g. [1], [2] or [16]) :

$$L_t = L_t^c + L_t^d = bt + W_t + \int_{|x| < 1} x \tilde{N}(t, dx) + \int_{|x| \geq 1} x N(t, dx),$$

where  $W$  is a Brownian motion,  $N$  is an independent Poisson random measure on  $\mathbb{R}^+ \times (\mathbb{R}^d \setminus \{0\})$  with levy measure  $\nu$ , and  $\tilde{N}$  is the compensated Poisson random measure. Then we can write  $B_T^{t,\theta}$  in the following form :

$$B_T^{t,\theta} = \int_t^T {}^1\theta^c b ds + \int_t^T {}^2\theta^c dW_s + \int_t^T \int_{|x| < 1} \theta_s^d x \tilde{N}(ds, dx) + \int_t^T \int_{|x| \geq 1} \theta_s^d x N(ds, dx).$$

Firstly, we will suppose that  $\nu(\mathbb{R}^d) < \infty$  which means that  $L_t$  has finite activity (i.e. it has a finite number of jumps in any finite period of time).

Let  $B_T^{t,\theta,c}$  and  $B_T^{t,\theta,d}$  be the continuous part and the discontinuous part of  $B_T^{t,\theta}$  defined respectively by

$$B_T^{t,\theta,c} = \int_t^T {}^1\theta_s^c b ds + \int_t^T {}^2\theta_s^c dW_s$$

and

$$B_T^{t,\theta,d} = \int_t^T \int_{|x|<1} \theta_s^d x \tilde{N}(ds, dx) + \int_t^T \int_{|x|\geq 1} \theta_s^d x N(ds, dx).$$

Let  $\psi \in C_b^{2,3}((0, T) \times \mathbb{R}^d)$  be such that  $\psi \geq v$  and, for a fixed  $(t, x) \in (0, T) \times \mathbb{R}^d$ ,  $\psi(t, x) = v(t, x)$ . By Itô's formula for Lévy-type stochastic integral, it follows that

$$\begin{aligned} & \psi(t+h, x+B_{t+h}^{t,\theta}) - \psi(t, x) \\ &= \int_t^{t+h} \frac{\partial \psi}{\partial s}(s, x+B_s^{t,\theta}) ds + \frac{1}{2} \int_t^{t+h} \text{tr}[\theta_s^c {}^2\theta_s^c D^2 \psi](s, x+B_{s-}^{t,\theta}) ds + \int_t^{t+h} \langle D\psi(s, x+B_{s-}^{t,\theta}), dB_s^{t,\theta,c} \rangle \\ &+ \int_t^{t+h} \int_{|z|\geq 1} [\psi(s, x+B_{s-}^{t,\theta} + \theta_s^d z) - \psi(s, x+B_{s-}^{t,\theta})] N(ds, dz) \\ &+ \int_t^{t+h} \int_{|z|<1} [\psi(s, x+B_{s-}^{t,\theta} + \theta_s^d z) - \psi(s, x+B_{s-}^{t,\theta})] \tilde{N}(ds, dz) \\ &+ \int_t^{t+h} \int_{|z|<1} [\psi(s, x+B_{s-}^{t,\theta} + \theta_s^d z) - \psi(s, x+B_{s-}^{t,\theta}) - \langle \theta_s^d z, D\psi(s, x+B_{s-}^{t,\theta}) \rangle] ds \nu(dz) \\ &\triangleq \int_t^{t+h} I_1 ds + \int_t^{t+h} \langle D\psi(s, x+B_{s-}^{t,\theta}), {}^2\theta_s^c dW_s \rangle \\ &+ \int_t^{t+h} \int_{|z|\geq 1} I_2 N(ds, dz) + \int_t^{t+h} \int_{|z|<1} I_2 \tilde{N}(ds, dz) - \int_t^{t+h} \int_{|z|<1} I_3 ds \nu(dz). \end{aligned}$$

Obviously, we have  $E_P[\int_t^{t+h} \langle D\psi(s, x+B_{s-}^{t,\theta}), {}^2\theta_s^c dW_s \rangle] = 0$ .

The uniformly Lipschitz continuity of  $(\frac{\partial \psi}{\partial s} + \langle D\psi, {}^1\theta_s^c b \rangle + \frac{1}{2} \text{tr}[\theta_s^c {}^2\theta_s^c D^2 \psi])(s, y)$  in  $(s, y)$  yields

$$\begin{aligned} E_P[I_1] &= E_P\left[\frac{\partial \psi}{\partial s} + \langle D\psi, {}^1\theta_s^c b \rangle + \frac{1}{2} \text{tr}[\theta_s^c {}^2\theta_s^c D^2 \psi](s, x+B_{s-}^{t,\theta})\right] \\ &\leq E_P\left[\frac{\partial \psi}{\partial s} + \langle D\psi, {}^1\theta_s^c b \rangle + \frac{1}{2} \text{tr}[\theta_s^c {}^2\theta_s^c D^2 \psi](t, x) + C_1(h+h^{1/2})\right], \\ E_P\left[\int_t^{t+h} \int_{|z|\geq 1} I_2 N(ds, dz)\right] \\ &\leq E_P\left[\int_t^{t+h} \int_{|z|\geq 1} (\psi(t, x+\theta_s^d z) - \psi(t, x) + C_1(h+h^{1/2})) N(ds, dz)\right] \end{aligned} \tag{7}$$

and

$$\begin{aligned} & E_P\left[\int_t^{t+h} \int_{|z|<1} \langle \theta_s^d z, D\psi(s, x+B_{s-}^{t,\theta}) \rangle ds \nu(dz)\right] \\ &\leq E_P\left[\int_t^{t+h} \int_{|z|<1} (\langle \theta_s^d z, D\psi(t, x) \rangle + C_1(h+h^{1/2})|\theta_s^d z|) ds \nu(dz)\right]. \end{aligned} \tag{8}$$

Since  $\theta_s$  is a predictable process, then it is independent of  $N(ds, dz)$  and  $\tilde{N}(ds, dz)$ , henceforth we have

$$E_P \left[ \int_t^{t+h} \int_{|z|<1} I_2 \tilde{N}(ds, dz) \right] = 0$$

and

$$\begin{aligned} & E_P \left[ \int_t^{t+h} \int_{|z|\geq 1} (\psi(t, x + \theta_s^d z) - \psi(t, x)) N(ds, dz) \right] \\ &= \int_t^{t+h} \int_{|z|\geq 1} E_P(\psi(t, x + \theta_s^d z) - \psi(t, x)) \nu(dz) ds. \end{aligned}$$

Thus, by the assumption on  $\nu$  and the inequalities (7) and (8), we have

$$\begin{aligned} & E_P \left[ \int_t^{t+h} \int_{|z|\geq 1} I_2 N(ds, dz) - \int_t^{t+h} \int_{|z|<1} I_3 ds \nu(dz) \right] \\ &\leq \int_t^{t+h} \int_{\mathbb{R}^d \setminus \{0\}} E_P(\psi(t, x + \theta_s^d z) - \psi(t, x)) \nu(dz) ds - E_P \left[ \int_t^{t+h} \int_{|z|<1} \langle D\psi(t, x), \theta_s^d z \rangle \nu(dz) ds \right] \\ &\quad + C_2(h^2 + h^{3/2}) \\ &\leq \int_t^{t+h} \int_{\mathbb{R}^d \setminus \{0\}} \int_{\mathbb{R}^d \setminus \{0\}} (\psi(t, x + z') - \psi(t, x)) F_{\theta_s^d}(d(\frac{z'}{z})) \nu(dz) ds + E_P \left[ \int_t^{t+h} \langle D\psi(t, x), \theta_s^d b' \rangle ds \right] \\ &\quad + C_2(h^2 + h^{3/2}) \\ &= h \int_{\mathbb{R}^d \setminus \{0\}} (\psi(t, x + z') - \psi(t, x)) \nu'(dz') + E_P \left[ \int_t^{t+h} \langle D\psi(t, x), \theta_s^d b' \rangle ds \right] \\ &\quad + C_2(h^2 + h^{3/2}), \end{aligned}$$

where  $F_{\theta_s^d}$  is the probability distribution function of  $\theta_s^d$ , and we denote  $\nu'(dz') = \int_{\mathbb{R}^d \setminus \{0\}} F_{\theta_s^d}(d(\frac{z'}{z})) \nu(dz)$ , and  $b' = - \int_{|z|<1} z \nu(dz)$ .

Hence, from the dynamic programming principle (5) it follows that

$$\begin{aligned} 0 &= \sup_{\theta \in \mathcal{A}_{t,t+h}^\Theta} E_P[v(t+h, x + B_{t+h}^{t,\theta}) - v(t, x)] \\ &\leq \sup_{\theta \in \mathcal{A}_{t,t+h}^\Theta} E_P[\psi(t+h, x + B_{t+h}^{t,\theta}) - \psi(t, x)] \\ &\leq \sup_{\theta \in \mathcal{A}_{t,t+h}^\Theta} \int_t^{t+h} E_P \left[ \frac{\partial \psi}{\partial s} + \langle D\psi, {}^1\theta_s^c b + \theta_s^d b' \rangle + \frac{1}{2} \text{tr} [{}^2\theta_s^c {}^2\theta_s^c{}^T D^2 \psi] \right] (t, x) ds + C_1(h^2 + h^{3/2}) \\ &\quad + \sup_{\nu' \in \mathcal{V}} h \int_{\mathbb{R}^d \setminus \{0\}} (\psi(t, x + z') - \psi(t, x)) \nu'(dz') + C_2(h^2 + h^{3/2}) \\ &\leq h \sup_{(\nu', q, \gamma) \in \mathcal{U}} \left\{ \int_{\mathbb{R}^d \setminus \{0\}} (\psi(t, x + z) - \psi(t, x)) \nu'(dz) + \langle D\psi(t, x), q \rangle + \frac{1}{2} \text{tr} [\gamma \gamma^T D^2 \psi(t, x)] \right\} \\ &\quad + h \frac{\partial \psi}{\partial s}(t, x) + (C_1 + C_2)(h^2 + h^{3/2}). \end{aligned}$$

Consequently,

$$\frac{\partial \psi}{\partial s}(t, x) + \sup_{(\nu', q, \gamma) \in \mathcal{U}} \left\{ \int_{\mathbb{R}^d \setminus \{0\}} (\psi(t, x+z) - \psi(t, x)) \nu'(dz) + \langle D\psi(t, x), q \rangle + \frac{1}{2} \text{tr}[\gamma \gamma^T D^2 \psi(t, x)] \right\} \geq 0.$$

By the definition,  $v$  is a viscosity subsolution. Now we will prove that this conclusion is also true if we remove the condition  $\nu(\mathbb{R}^d) < \infty$ .

Denote

$${}^\varepsilon L_t = bt + W_t + \int_{\varepsilon \leq |x| < 1} x \tilde{N}(t, dx) + \int_{|x| \geq 1} x N(t, dx)$$

and

$${}^\varepsilon B_T^t = B_T^{t, \theta, c} + {}^\varepsilon B_T^{t, \theta, d},$$

where

$${}^\varepsilon B_T^{t, \theta, d} = \int_t^T \int_{\varepsilon \leq |x| < 1} \theta_s^d x \tilde{N}(ds, dx) + \int_t^T \int_{|x| \geq 1} \theta_s^d x N(ds, dx),$$

note that each  $({}^\varepsilon L_t)_{t \geq 0}$  is a compound poisson process, hence is a finite-activity process.

$$\begin{aligned} 0 &= \sup_{\theta \in \mathcal{A}_{t, t+h}^\Theta} E_P[v(t+h, x + B_{t+h}^{t, \theta}) - v(t, x)] \\ &\leq \sup_{\theta \in \mathcal{A}_{t, t+h}^\Theta} E_P[v(t+h, x + {}^\varepsilon B_{t+h}^{t, \theta}) - v(t, x)] + C \sup_{\theta \in \mathcal{A}_{t, t+h}^\Theta} E_P\left[\left| \int_t^{t+h} \int_{|x| < \varepsilon} \theta_s^d z \tilde{N}(ds, dz) \right|\right] \\ &\leq \sup_{\theta \in \mathcal{A}_{t, t+h}^\Theta} E_P[\psi(t+h, x + {}^\varepsilon B_{t+h}^{t, \theta}) - \psi(t, x)] + C \sup_{\theta \in \mathcal{A}_{t, t+h}^\Theta} \int_t^{t+h} \int_{|x| < \varepsilon} E_P[|\theta_s^d z|] \nu(dz) ds \\ &\leq \sup_{\theta \in \mathcal{A}_{t, t+h}^\Theta} \int_t^{t+h} E_P\left[\frac{\partial \psi}{\partial s} + \langle D\psi, {}^1\theta_s^c b + \theta_s^d b' \rangle + \frac{1}{2} \text{tr}[\theta_s^{c2} \theta_s^{cT} D^2 \psi]\right](t, x) ds + C_1(h^2 + h^{3/2}) \\ &\quad + \sup_{\theta \in \mathcal{A}_{t, t+h}^\Theta} \int_t^{t+h} \int_{|z| \geq \varepsilon} E_P(\psi(t, x + \theta_s^d z) - \psi(t, x)) \nu(dz) ds + C_2(h^2 + h^{3/2}) + C_3 h \int_{|z| < \varepsilon} |z| \nu(dz), \end{aligned}$$

where  $b' = \int_{\varepsilon \leq |z|} |z| \nu(dz)$ . Here note that  $C_2$  is a finite number for each fixed  $\varepsilon$ . Since  $h > 0$  is arbitrary small, and  $\int_{\mathbb{R}^d \setminus \{0\}} |z| \nu(dz) < \infty$  implies that  $\lim_{\varepsilon \downarrow 0} \int_{|z| < \varepsilon} |z| \nu(dz) = 0$ . Then we divide both sides of last inequality by  $h$ , and let firstly  $h$  and then  $\varepsilon$  go to zero, we could finally obtain the result.

Similarly, we can prove that  $v$  is also a supersolution. The proof is complete now.  $\square$

We know that  $u(t, x) := v(T - t, x)$ , then  $u$  is a viscosity solution of  $\frac{\partial u}{\partial t} - G(u, Du, D^2 u) = 0$ , with initial condition  $u(0, x) = \varphi(x)$ . From the uniqueness of the viscosity solution of integro-pde and Theorem 4.3, we get the following proposition :

**Proposition 4.4**

$$\begin{aligned} \mathbb{E}^G[\varphi(X_{t_1}, X_{t_2} - X_{t_1}, \dots, X_{t_n} - X_{t_{n-1}})] &= \sup_{\theta \in \mathcal{A}_{0, \infty}^\Theta} E_P[\varphi(B_{t_1}^{0, \theta}, B_{t_2}^{t_1, \theta}, \dots, B_{t_n}^{t_{n-1}, \theta})] \\ &= \sup_{\theta \in \mathcal{A}_{0, \infty}^\Theta} E_{P_\theta}[\varphi(B_{t_1}^0, B_{t_2}^{t_1}, \dots, B_{t_n}^{t_{n-1}})], \end{aligned}$$

where  $P_\theta$  is the law of the process  $B_t^{0, \theta}$ ,  $t \geq 0$ , for  $\theta \in \mathcal{A}_{0, \infty}^\Theta$ .

**Proposition 4.5** The family of probability measures  $\{P_\theta, \theta \in \mathcal{A}_{0, \infty}^\Theta\}$  on  $\Omega = D_0([0, \infty), \mathbb{R}^d)$  is tight.

*Proof.* Since for any  $s, u, t$   $0 \leq s \leq u \leq t < \infty$ , there exists  $C' > 0$ , such that  $\mathbb{E}^G[|X_t^c - X_s^c|^4] \leq C'|t - s|^2$ , (see. Proposition 49 of [6]) and

$$\begin{aligned}\mathbb{E}^G[|X_t^d - X_u^d| \cdot |X_u^d - X_s^d|] &= \sup_{\theta \in \mathcal{A}_{t,T}^{\Theta}} E_{P_\theta}[|B_t^{u,\theta,d}| \cdot |B_u^{s,\theta,d}|] \\ &\leq C'|t - s|^2.\end{aligned}$$

Therefore the statement follows from Corollary A.2 and Theorem A.3.  $\square$

**Example 1** Let  $(L_t)_{t \geq 0}$  be a homogeneous Poisson process with intensity  $\lambda = 1$  denoted by  $(N_t)_{t \geq 0}$ , then we take especially a collection of  $\mathbb{F}$ -predictable process  $\mathcal{A}_{t,T}^{\Theta^d}$ , where  $\Theta^d = \{0, 1\}$  and for each  $t > 0$ ,  $\theta_t^d$  follows a Bernoulli distribution with success probability  $p$ , where  $p \in [\lambda, 1]$ . If  $(X_t)_{t \geq 0}$  is the 1-dimensional  $G$ -Poisson process [10] defined by the following equation :

$$\partial_t u(t, x) - G_\lambda(u(t, x+1) - u(t, x)) = 0, \quad u(0, x) = \varphi(x),$$

where  $G_\lambda(a) = a^+ - \lambda a^-$ ,  $\lambda \in [0, 1]$ . It is easy to check that, for each  $n \in \mathbb{N}$ , any  $\varphi \in C_{b,Lip}(\mathbb{R}^{d \times n})$  and  $0 \leq t_1, \dots, t_n < \infty$ ,

$$\mathbb{E}^G[\varphi(X_{t_1}, X_{t_2} - X_{t_1}, \dots, X_{t_n} - X_{t_{n-1}})] = \sup_{\theta^d \in \mathcal{A}_{0,\infty}^{\Theta^d}} E_P[\varphi(\int_{t_1}^0 \theta_s^d dN_s, \int_{t_2}^{t_1} \theta_s^d dN_s, \dots, \int_{t_n}^{t_{n-1}} \theta_s^d dN_s)].$$

## 5. Characterization of $L_{ip}(\Omega)$

In [6], we know that if  $\Omega_T = C_0([0, T], \mathbb{R}^d)$  for  $T > 0$ , (resp.  $\Omega = C_0([0, \infty), \mathbb{R}^d)$ ), then for  $t \leq T$ ,

$$L_{ip}(\Omega_t) \subseteq L_{ip}(\Omega_T) \subset C_b(\Omega_T) \quad (\text{resp. } L_{ip}(\Omega) \subset C_b(\Omega)),$$

and that an element  $Y$  of the space  $L_G^p(\Omega)$  is a quasi-continuous function  $Y = Y(\omega)$  defined on  $\Omega$ .  $L_G^p(\Omega)$  is also proved to be identified with the space  $\mathcal{L}^p$  that introduced in [7].

But if  $\Omega = D_0([0, T], \mathbb{R}^d)$ , this relationship of inclusion between  $L_{ip}(\Omega_T)$  and  $C_b(\Omega_T)$  is no longer true. In fact, for any fixed  $t \in [0, T]$ , the variable  $B_t(\omega) = \omega_t$  (or  $\omega(t)$ ) on  $(\Omega, \mathcal{B}(\Omega))$

$$B_t : \Omega \mapsto \mathbb{R}$$

$$\omega \mapsto \omega_t$$

is continuous in  $\omega$  if and only if  $\omega$  is continuous at  $t$  (see Section 12 of [3]), thus  $L_{ip}(\Omega_T)$  does not belong to  $C_b(\Omega_T)$ .

**Proposition 5.1** Let  $\Omega_T = D_0([0, T], \mathbb{R})$  be equipped with the Skorohod metric  $d_s$ , then  $B_t$  is either upper semi-continuous (in short, u.s.c.) or lower semi-continuous (in short, l.s.c.) at each point  $\omega \in \Omega_T$ .

*Proof.* For any  $\omega_o \in \Omega_T$ ,  $\omega_o$  is either u.s.c. or l.s.c. at each fixed  $t \in [0, T]$ . Without loss of generality, we will only consider the u.s.c. case. Then for any  $\varepsilon > 0$ , there exists  $\delta > 0$ , such that for any  $s \in [0, T]$  satisfying that  $|s - t| < \delta$ , we have  $\omega_o(s) < \omega_o(t) + \varepsilon/2$ . Now we can choose  $\eta = \min(\varepsilon/2, \delta) > 0$ , for any  $\omega$  such that  $d_s(\omega, \omega_o) < \eta$ , there exists some  $\lambda$  which is a strictly increasing, continuous mapping from  $[0, T]$  onto itself with  $\lambda 0 = 0$  and  $\lambda T = T$ , then we have  $\sup_t |\omega(t) - \omega_o(\lambda(t))| < \eta$ , and  $\sup_t |\lambda(t) - t| < \eta$ . Hence we get

$$\omega(t) - \omega_o(t) = \omega(t) - \omega_o(\lambda(t)) + \omega_o(\lambda(t)) - \omega_o(t) < \eta + \varepsilon/2 < \varepsilon.$$

Thus the proof is complete.  $\square$

**Proposition 5.2** For each  $X \in L_{ip}(\Omega)$  and  $\varepsilon > 0$ , there exists  $Y \in C_b(\Omega)$  such that  $\mathbb{E}^G[|X - Y|] < \varepsilon$ .

*Proof.* Without loss of generality, we suppose that  $\Omega = D_0([0, \infty), \mathbb{R})$ . Let each  $X \in L_{ip}(\Omega)$  be with the form  $X = \varphi(B_{t_1}, B_{t_2}, \dots, B_{t_m})$ , where  $\varphi \in C_{b, Lip}(\mathbb{R}^m)$  and  $0 \leq t_1 \leq t_2 \leq \dots \leq t_m < \infty$ . For each  $i = 1, 2, \dots, m$ , let  $h_n^i(\omega) = n \int_{t_i}^{t_i + \frac{1}{n}} B_s(\omega) ds$ , then  $h_n^i$  is continuous in the Skorohod topology (see [3]). In fact, if  $\omega_k \rightarrow \omega$  in the Skorohod topology, then  $\omega_k(s) \rightarrow \omega(s)$  for continuity points  $s$  and hence for points  $s$  outside a set of Lebesgue measure 0; since  $\omega_k$  are uniformly bounded, we have  $\lim_k h_n^i(\omega_k) \rightarrow h_n^i(\omega)$ . By the right continuity of  $\omega$ ,  $h_n^i(\omega) \rightarrow B_{t_i}(\omega)$  as  $n \rightarrow \infty$ . Then it follows that

$$\begin{aligned} \mathbb{E}^G[|\varphi(B_{t_1}, B_{t_2}, \dots, B_{t_m}) - \varphi(h_n^1, h_n^2, \dots, h_n^m)|] &\leq \sum_{i=1}^m \mu \mathbb{E}^G[|B_{t_i} - h_n^i|] \\ &= \sum_{i=1}^m \mu \mathbb{E}^G[|B_{t_i} - n \int_{t_i}^{t_i + \frac{1}{n}} B_s ds|] \\ &= \sum_{i=1}^m \mu \mathbb{E}^G[|n \int_{t_i}^{t_i + \frac{1}{n}} (B_{t_i} - B_s) ds|] \\ &\leq \sum_{i=1}^m \mu n \int_{t_i}^{t_i + \frac{1}{n}} \mathbb{E}^G[|B_{t_i} - B_s|] ds \\ &\leq \sum_{i=1}^m \mu n \int_{t_i}^{t_i + \frac{1}{n}} C(|t_i - s|^{1/2} + |t_i - s|) ds \end{aligned}$$

where  $\mu$  is the Lipschitz constant of  $\varphi$ . Hence for each positive  $\varepsilon$ , we can choose some  $n_0 > 0$  and set  $Y = \varphi(h_{n_0}^1, h_{n_0}^2, \dots, h_{n_0}^m)$  such that  $\mathbb{E}^G[|X - Y|] < \varepsilon$ .  $\square$

**Remark 5** This proposition implies that  $L_{ip}(\Omega) \subseteq \mathbb{L}_c^1$ , hence  $L_G^1(\Omega) \subseteq \mathbb{L}_c^1$ .

As shown in previous sections, we use two different methods to prove that  $\mathbb{E}^G$  is an upper expectation associated to a weakly relatively compact family  $\mathcal{P}_1$  and now denote by  $\mathcal{P} = \overline{\mathcal{P}_1}$  the closure of  $\mathcal{P}_1$  under the topology of weak convergence, then  $\mathcal{P}$  is weakly compact. For each  $X \in L^0(\Omega)$  such that  $E_P[X]$  exists for each  $P \in \mathcal{P}$ , we set

$$\mathbb{E}^{\mathcal{P}}[X] = \sup_{P \in \mathcal{P}} E_P[X]$$

and

$$\mathbb{E}^{\mathcal{P}_1}[X] = \sup_{P \in \mathcal{P}_1} E_P[X].$$

Proposition 5.2 together with Proposition 2.3 yields the following theorem.

**Theorem 5.1** For each  $X \in L_G^1(\Omega)$ , we have  $\mathbb{E}^G[X] = \mathbb{E}^{\mathcal{P}}[X] = \mathbb{E}^{\mathcal{P}_1}[X]$ .

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## Appendix A

**Lemma A.1** (See [3]) Let  $\{P_n\}$  be a sequence of probability measures on a measurable space  $(D_0([0, T], \mathbb{R}), \mathcal{D})$  (resp.  $(D_0([0, \infty), \mathbb{R}), \mathcal{D}_\infty)$ ), then  $\{P_n\}$  is tight if and only if these two conditions hold:  
(i) For each  $t$  in a set  $\mathcal{T}$  dense in  $[0, T]$  and contains  $T$  (resp. dense in  $[0, \infty)$ ),

$$\lim_{a \rightarrow \infty} \limsup_n P_n[\omega : |\omega_t| \geq a] = 0.$$

(ii) For each positive  $\varepsilon$  (resp. for each positive  $\varepsilon$  and  $T$ ),

$$\begin{cases} \lim_{\delta \rightarrow 0} \limsup_n P_n[\omega : V_T''(\omega, \delta) \geq \varepsilon] = 0. \\ \lim_{\delta \rightarrow 0} \limsup_n P_n[\omega : |\omega_\delta| \geq \varepsilon] = 0. \\ \lim_{\delta \rightarrow 0} \limsup_n P_n[\omega : |\omega_{T-} - \omega_{T-\delta}| \geq \varepsilon] = 0. \end{cases}$$

where  $V_T''(\omega, \delta) = \sup_{s \leq u \leq t, t-s \leq \delta} \{|\omega_u - \omega_s| \wedge |\omega_t - \omega_u|\}$ , the supremum extending over all triples  $s, u, t$  in  $[0, T]$  satisfying the constraints.

**Corollary A.2** Let  $X$  and  $Y$  be two càdlàg stochastic processes. Assume that the distributions of  $X$  and  $Y$  respectively under the probability measures  $(P_n, n \in \mathbb{N})$  denoted respectively by  $(P_n(X \in \cdot), n \in \mathbb{N})$  and  $(P_n(Y \in \cdot), n \in \mathbb{N})$  are both tight, then  $(P_n(X + Y \in \cdot), n \in \mathbb{N})$  is tight.

*Proof.* Apply the triangle inequality in order to check the conditions in the preceding lemma.  $\square$

**Theorem A.3** (Kolmogorov-Chentsov's criterion for weak relative compactness) Let  $\mathcal{P}$  be any subset of the collection of all probability measures on  $D_0([0, T], \mathbb{R})$  and  $\mathbb{E}$  the upper expectation related to  $\mathcal{P}$ . If the following conditions are satisfied:

- (i)  $\exists a > 0$ , such that  $\mathbb{E}[|\omega_t - \omega_s|] \leq C|t - s|^a, \forall t, s \in [0, T]$ ;
- (ii)  $\mathbb{E}[|\omega_t - \omega_u|^p |\omega_u - \omega_s|^q] \leq C|t - s|^{1+r}$ , for some  $C, r > 0, p, q \geq 0$  with  $p + q > 0$ , and all  $0 \leq s \leq u \leq t \leq T$ , then  $\mathcal{P}$  is relatively compact.

*Proof.* Let  $\{P_n\}$  be any sequence in  $\mathcal{P}$ . We check the conditions of the previous lemma. Obviously, condition (i) implies  $\mathbb{E}[|\omega_\delta|] \leq C\delta^a$  and  $\mathbb{E}[|\omega_{T-} - \omega_{T-\delta}|] \leq C\delta^a$ . By condition(ii),  $\forall \eta > 0, \alpha \in (0, \frac{r}{p+q})$ ,  $\exists K > 0$ , for every  $n$ , we have

$$P_n[|\omega_t - \omega_u| \geq K|t - s|^\alpha, |\omega_u - \omega_s| \geq K|t - s|^\alpha, \forall 0 \leq s \leq u \leq t \leq T] \leq \eta,$$

This obviously implies  $\limsup_n P_n[V_T''(\omega, \delta) \geq \varepsilon] \leq \eta$  with  $\delta = \left|\frac{\varepsilon}{K}\right|^{1/\alpha}$ . Hence by Lemma A.1, we get the conclusion.  $\square$

## References

- [1] Applebaum, D. (2004): *Lévy processes and Stochastic Calculus*, Cambridge University Press.
- [2] Bertoin, J. (1996): *Lévy processes*, Cambridge University Press.
- [3] Billingsley, P. (1999): *Convergence of probability measures* (second edition), John Wiley & Sons.
- [4] Choquet, G. (1955): *Theory of capacities*, Ann. Inst. Fourier 5, 131-295.
- [5] Dellacherie, C. (1972): *Capacité et Processus Stochastiques*, Springer Verlag, Berlin.
- [6] Denis, L., Hu, M., Peng, S. (2008): *Function spaces and capacity related to a Sublinear Expectation: application to G-Brownian Motion Pathes*, arXiv:0802.1240v1 [math.PR] 9 Feb 2008.
- [7] Denis, L., Martini, C. (2006): A Theoretical Framework for the Pricing of Contingent Claims in the Presence of Model Uncertainty, The Annals of Applied Probability, vol.16, No.2, pp 827-852.
- [8] Gikhman, I.I., Skorohod, A.N. (1969): *Introduction to the Theory of Stochastic Processes*, W.B. Saunders, Philadelphia.

- [9] Hu, M., Peng, S. (2009a): *G-Lévy processes under sublinear expectations*, arXiv:0911.3533v1 [math.PR]
- [10] Hu, M., Peng, S. (2009b): *On Representation Theorem of G-Expectations and Paths of G-Brownian Motion*, Acta Mathematicae Applicatae Sinica, English Series, 25(3), 539-546.
- [11] Huber, P., Strassen, V. (1973): *Minimax tests and the Neyman-Pearson Lemma for capacity*, The Annals of Statistics, Vol.1, No.2 252-263.
- [12] Peng, S. (2006): *G-Expectation, G-Brownian Motion and Related Stochastic Calculus of Itô Type*, arXiv:math.PR/0601035v2.
- [13] Peng, S. (2007a): *G-Brownian Motion and Dynamic Risk Measure under Volatility Uncertainty*, arXiv:0711.2834v1 [math.PR]. 19 Nov 2007.
- [14] Peng, S. (2008a): *Multi-dimensional G-Brownian Motion and Related stochastic Calculus under G-expectation*, Stochastic Processes and Their Applications, Volume 118, Pages 2223–2253.
- [15] Peng, S. (2008b): *A New Central Limit Theorem under Sublinear Expectations*, arXiv:0803.2656v1 arXiv:0803.2656v1 [math.PR] 18 Mar 2008.
- [16] Sato, K.-I. (1999): *Lévy processes and Infinitely Divisible Distributions*, Cambridge University Press.
- [17] Soner, H.M., Touzi, N., Zhang, J. (2011): *Martingale Representation Theorem for the G-expectation*, Stochastic Processes and Their Applications, Volume 121, Pages 265-287.